THE MAGNETIC FIELD IN INHOMOGENEOUS TURBULENT FLOW

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We consider a particular model of magnetohydrodynamic turbulents. The most fundamental assumption we make is that the velocity correlation time is negligible. By using a selective summation of the perturbation theory series an exact equation for the magnetic field is obtained when the mean square value of the velocity depends on coordinates, i.e., when the turbulence isinhomogeneous. The result makes it possible to obtain the "macroscopic" Maxwell's equations, i.e., the equations for the large-scale components of the electromagnetic field.

In problems of magnetohydrodynamic turbulence the turbulence is usually considered homogeneous. The results of such considerations reduce basically to the following. If a weak large-scale magnetic field is impressed on a high-conductivity turbulent fluid (the scale of the field being much larger than the scale of the pulsations) in the absence of gyrotropy there is an anomalous diffusion of the field [1].

Actual turbulence is always inhomogeneous. For example, there is always a boundary of the turbulence. It would seem that an inhomogeneity in the intensity of the pulsations would lead simply to anomalous diffusion with a diffusion coefficient depending on coordinates. In this case macroscopic electrodynamics, i.e., the equations for the large-scale fields, would not differ from the "microscopic" or ordinary Maxwell's equations, but in Ohm's law the ordinary electrical conductivity would be replaced by an anomalous conductivity depending on coordinates. The fact is that the inhomogeneity gives rise to a new effect analogous to diamagnetism. This was first noted by Ya. B. Zel'dovich [2] for the idealized two-dimensional case and by Radler [3] for a weakly conducting fluid.

We consider a high-conductivity fluid. Since pulsations of the magnetic field in this case are not small in comparison with the large-scale field, the velocity perturbation theory series cannot be broken off as is done in [3]. In using selective summation of the series we neglect the correlation time of the pulsations. This is justified since it will be shown below that the characteristic time of variation of the magnetic field is appreciably longer than the correlation time.

We have the familiar equation for the magnetic field **H**

$$\frac{\partial \mathbf{H}}{\partial t} = \operatorname{rot}\,[\mathbf{v}, \mathbf{H}] + \nu_m \Delta \mathbf{H}$$

where **v** is the velocity and ν_m is the magnetic viscosity. We assume that the velocity field is given so that the problem is purely kinematical. The equation of motion is not required. This procedure is possible if the energy of the large-scale magnetic field is less than the energy of the pulsations. We proceed to the specification of the velocity field.

1. Derivation of the Velocity Spectrum Tensor. We write $\mathbf{v}(\mathbf{x}, t)$ in the form

$$\mathbf{v}(\mathbf{x}, t) = f(\mathbf{x})\,\mathbf{u}(\mathbf{x}, t), \quad \langle u^2 \rangle = 1 \tag{1.1}$$

The symbol < > denotes averaging over the pulsations. This is possible if $\langle v^2 \rangle$ does not depend on time, which we assume is the case. We also assume that the inhomogeneity is weak, i.e., that $f(\mathbf{x})$ varies

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slowly with the correlation length. Then one cannot expect that the velocity field will be isotropic since ∇f determines a preferred direction. Strictly speaking, the direction of the large-scale field can also be singled out, but since the energy of this field is small, we neglect its effect on the motion. Then the correlation tensor has the form [4]

$$\langle v_i(\mathbf{x}_1, t)\, v_j(\mathbf{x}_2, t)\rangle = f(\mathbf{x}_1)\, f(\mathbf{x}_2)\left(A\delta_{ij} + Br_ir_j + C_1r_i\frac{\partial\varphi}{\partial x_j} + C_2r_j\frac{\partial\varphi}{\partial x_i}\right)$$

$$(\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1,\ \ \varphi(\mathbf{x}) = f^2(\mathbf{x})) \tag{1.2}$$

Here A, B, C_1, and C_2 are functions of r and $(\mathbf{r}\nabla\varphi)$. Since the inhomogeneity is weak, we set

$$f(\mathbf{x}_1)\, f(\mathbf{x}_2) = \varphi(\mathbf{x}_1) + {}^1/_2(\mathbf{r}\nabla\varphi)$$

$$A = A_1(r) + (\mathbf{r}\nabla\varphi)A_2(r),\ \ C_1 = C_3(r) + (\mathbf{r}\nabla\varphi)C_4(r)$$

$$B = B_1(r) + (\mathbf{r}\nabla\varphi)B_2(r),\ \ C_2 = C_5(r) + (\mathbf{r}\nabla\varphi)C_6(r) \tag{1.3}$$

We substitute (1.3) into (1.2) and require that

$$\langle v_i(\mathbf{x}_1)\, v_j(\mathbf{x}_2)\rangle = \langle v_j(\mathbf{x}_2)\, v_i(\mathbf{x}_1)\rangle$$

retaining first-order terms in the small quantity $r_i\partial\varphi/\partial x_j$ in the tensor. Then (1.2) takes the form

$$\langle v_i(\mathbf{x}_1)\, v_j(\mathbf{x}_2)\rangle = \varphi(\mathbf{x}_1)\left[A_1\delta_{ij} + B_1r_ir_j + C_3\left(r_i\frac{\partial\varphi}{\partial x_j} - r_j\frac{\partial\varphi}{\partial x_i}\right)\right] + \frac{1}{2}(\mathbf{r}\nabla\varphi)\,[A_1\delta_{ij} + B_1r_ir_j] \tag{1.4}$$

We go over to the Fourier representation of (1.4)

$$\mathbf{v}(\mathbf{x}) = \int \mathbf{u}(\mathbf{k}) \exp i(\mathbf{kx})\, d\mathbf{k},\quad \varphi(\mathbf{r}) = \int \varphi(\mathbf{k}) \exp i(\mathbf{kr})\, d\mathbf{k}$$

$$\langle v_i(\mathbf{x}_1)\, v_j(\mathbf{x}_2)\rangle = \int \langle u_i(\mathbf{k}_1)\, u_j(\mathbf{k}_2)\rangle \exp i\,[(\mathbf{k}_1\mathbf{x}_1) + (\mathbf{k}_2\mathbf{x}_2)]\, d\mathbf{k}_1 d\mathbf{k}_2 = \int f_{ij}(\mathbf{k}_1, \mathbf{k}) \exp i\,[(\mathbf{k}_1\mathbf{x}_1) + (\mathbf{kr})]\, d\mathbf{k}_1 d\mathbf{k}$$

$$f_{ij}(\mathbf{k}_1, \mathbf{k}) = \varphi(\mathbf{k}_1)\left[\left(A(k) - \frac{(\mathbf{kk}_1)}{2k}\frac{dA}{dk}\right)k_ik_j - \frac{1}{2}A(k)\,(k_ik_{1j} + k_jk_{1i})\right.$$

$$\left. + \left(B - \frac{(\mathbf{kk}_1)}{2k}\frac{dB}{dk}\right)\delta_{ij}\right] + C(k)\int \varphi(\mathbf{k}_1 - \mathbf{k}_3)\,\varphi(\mathbf{k}_3)\,[k_ik_{3j} - k_jk_{3i}]\, d\mathbf{k}_3$$

This tensor is considerably simplified if it is required to be solenoidal

$$k_if_{ij}(\mathbf{k}_1, \mathbf{k}) = 0,\quad (k_{1i} - k_i)\, f_{ij}(\mathbf{k}_1, \mathbf{k}) = 0 \tag{1.5}$$

The first of conditions (1.5) reduces f_{ij} to

$$f_{ij} = \varphi(\mathbf{k}_1)\left[\left(A(k) - \frac{(\mathbf{kk}_1)}{2k}\frac{dA}{dk}\right)(k^2\delta_{ij} - k_ik_j) - A(k)\,((\mathbf{k}_1\mathbf{k})\,\delta_{ij} - k_ik_{1j})\right] \tag{1.6}$$

The second of conditions (1.5) does not add anything new; it leads to

$$\varphi(\mathbf{k}_1)\frac{(\mathbf{k}_1\mathbf{k})}{2k}\frac{dA}{dk}\,(\mathbf{k}_1\mathbf{k})\, k_j - k^2k_{1j}) = 0$$

This equation is satisfied automatically, since it is assumed that $\varphi(\mathbf{r})$ varies slowly and that only first derivatives are taken into account; the last equation is quadratic in $\mathbf{k}_1$. We write out the expression for

$$f'_{ij}(\mathbf{k}_1, \mathbf{k}_2) = \langle u_i(\mathbf{k}_1)\, u_j(\mathbf{k}_2)\rangle = \varphi(\mathbf{k}_1 + \mathbf{k}_2)\,[A(k_2)\,(k_{2i}k_{1j} - (\mathbf{k}_1\mathbf{k}_2)\,\delta_{ij})$$

$$+ \frac{dA(k_2)}{dk_2}\frac{1}{2k_2}\,(k_2{}^2 + (\mathbf{k}_1\mathbf{k}_2))\,(k_{2i}k_{2j} - k_2{}^2\delta_{ij})] \tag{1.7}$$

If the velocity field is homogeneous (1.7) goes over into the well-known spectral tensor

$$\varphi(\mathbf{k}_1 + \mathbf{k}_2) = D\delta(\mathbf{k}_1 + \mathbf{k}_2)$$

$$f'_{ij}(\mathbf{k}_1, \mathbf{k}_2) = D\delta(\mathbf{k}_1 + \mathbf{k}_2)\, A(k_1)\,(k_1{}^2\delta_{ij} - k_{1i}k_{1j})$$

2. Derivation of the Equation for the Magnetic Field. In the following we use the velocity perturbation theory series. Let $\mathbf{H}(\mathbf{k}, t)$ be a Fourier component of the magnetic field

$$\mathbf{H}(\mathbf{k}, t) = \sum_{n=0}^{\infty} \mathbf{H}^{(n)}, \quad \mathbf{H}^{(0)} = \mathbf{H}(\mathbf{k}, 0) \exp(-\nu_m k^2 t)$$

$$\mathbf{H}^{(n+1)}(\mathbf{k}, t) = i \int_0^t \exp[k^2 \nu_m (t_1 - t)]\, dt_1 \int [\mathbf{k}\, [u(\mathbf{k} - \mathbf{k}', t_1)\, \mathbf{H}^{(n)}(\mathbf{k}', t_1)]]\, d\mathbf{k}' \tag{2.1}$$

Averaging over pulsations we separate out the large-scale component

$$\langle \mathbf{H}^{(n)} \rangle = \mathbf{B}^{(n)}, \quad \langle \mathbf{H} \rangle = \mathbf{B}$$

We use the following model of turbulence: 1) the velocity probability distribution is Gaussian; 2) we neglect the correlation time

$$\langle u_i(\mathbf{k}_1, t)\, u_j(\mathbf{k}_2, t') \rangle = m\delta(t - t')\, f'_{ij}(\mathbf{k}_1, \mathbf{k}_2)$$

Using this model we find that the odd terms of the series vanish and the even terms obey the recurrence relation

$$\mathbf{B}^{(2n)} = \frac{p}{2} \int_0^t \exp[\nu_m k^2 (t_1 - t)] \int d\mathbf{k}_1\, [\mathbf{k}\, [(\mathbf{k} - \mathbf{k}_1)\, \varphi(\mathbf{k} - \mathbf{k}_1)\, \mathbf{B}^{(2n-2)}(\mathbf{k}_1, t_1)]]\, dt_1 \tag{2.2}$$

$$+ p \int_0^t \exp[\nu_m k^2 (t_1 - t)] \int d\mathbf{k}_1\, [\mathbf{k}\, [\mathbf{k} - \mathbf{k}_1)\, \varphi(\mathbf{k}_1)\, \mathbf{B}^{(2n-2)}(\mathbf{k} - \mathbf{k}_1, t_1)]]\, dt_1$$

$$p = \frac{m}{3} \int A(k)\, k^2\, d\mathbf{k}$$

It is easy to show that the equation for which (2.2) is the perturbation theory series has the form

$$\frac{\partial \mathbf{B}(\mathbf{k}, t)}{\partial t} + \nu_m k^2 \mathbf{B}(\mathbf{k}, t) = \frac{p}{2} \int d\mathbf{k}_1\, [\mathbf{k}\, [(\mathbf{k} - \mathbf{k}_1)\, \varphi(\mathbf{k} - \mathbf{k}_1) \mathbf{B}(\mathbf{k}_1, t)]]$$
$$+ p \int d\mathbf{k}_1\, [\mathbf{k}\, [(\mathbf{k} - \mathbf{k}_1)\, \varphi(\mathbf{k}_1)\, \mathbf{B}(\mathbf{k} - \mathbf{k}_1, t)]] \tag{2.3}$$

or in r-space

$$\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} = \nu_m \Delta \mathbf{B} - \frac{p}{2} \operatorname{rot} [\nabla\varphi, \mathbf{B}] - p \operatorname{rot} \varphi \operatorname{rot} \mathbf{B} = -\operatorname{rot} \nu_m (1 + \chi)^{1/2} \operatorname{rot} (1 + \chi)^{1/2} \mathbf{B} \quad (\chi = p\varphi/\nu_m) \tag{2.4}$$

where χ is the magnetic Reynolds number. Equation (2.4) describes the diffusion of the large-scale field **B** in an inhomogeneous conductor with a variable electrical conductivity

$$\nu_{eff} = \nu_m (1 + \chi)^{1/2}, \quad \sigma_{eff} = \sigma (1 + \chi)^{-1/2} \tag{2.5}$$

and a variable magnetic permeability

$$\mu = (1 + \chi)^{-1/2} \tag{2.6}$$

In the most interesting case, $\chi \gg 1$, $\nu_{eff} \gg \nu_m$, $\sigma_{eff} \ll \sigma$, and $\mu \ll 1$. Now we can write the macroscopic Maxwell's equations

$$\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = -\operatorname{rot} \mathbf{E}, \quad \operatorname{div} \mathbf{E} = 4\pi\rho, \quad \operatorname{div} \mathbf{B} = 0 \tag{2.7}$$
$$\operatorname{rot} \mathbf{H} = \frac{4\pi}{c} \mathbf{j}, \quad \mathbf{E} = \langle \mathbf{e} \rangle, \quad \mathbf{H} = \frac{1}{\mu} \mathbf{B}$$

Here **e** is the intensity of the electric field.

Ohm's law takes the form

$$\mathbf{j} = \frac{\mathbf{E}\sigma}{(1+\chi)^{1/2}} \tag{2.8}$$

We write the expression for rot **B**, i.e., for the macroscopic current

$$\operatorname{rot} \mathbf{B} = \frac{4\pi}{c} \frac{\sigma \mathbf{E}}{1+\chi} - \frac{1}{2} \frac{[\nabla\chi, \mathbf{B}]}{1+\chi} \tag{2.9}$$

Equation (2.9) is identical with Radler's equation [3] obtained for small pulsations of the magnetic field by calculating the quadratic effect. The pulsations will be small if $\chi \ll 1$.

3. Boundary Value Problem. Suppose φ is constant in a certain region and falls to zero in the boundary layer of this region (turbulence in a bounded region). The boundary conditions for the current can be obtained by integrating (2.4) over the volume of the boundary layer, i.e., in the usual way. As a result, we obtain

$$(1+\chi) \operatorname{rot}_{t_1} \mathbf{B} = \operatorname{rot}_{t_2} \mathbf{B} \tag{3.1}$$

Here rot_{t_1} and rot_{t_2} denote the transverse components of rot **B** inside and outside the region, respectively. Of course (3.1) can be obtained from (2.9) also. We note that, as usual, $\mathbf{E}_t$ is continuous at the boundary. Naturally the normal component of the current also is continuous at the boundary.

The boundary conditions for **B** are

$$\mathbf{B}_{n_1} = \mathbf{B}_{n_2}, \quad (1+\chi)^{1/2} \mathbf{B}_{t_1} = \mathbf{B}'_{t_2} \tag{3.2}$$

The second condition of (3.2) follows from (2.6) and can be obtained by integrating (2.9) over the volume of the boundary layer. The surface currents are given by the second term on the right-hand side of (2.9).

Because of the variation of the electrical conductivity space charges are produced, or for an infinitely thin boundary layer, surface charges.

Taking the divergence of (2.9) and using (2.7) we obtain

$$\operatorname{div} \mathbf{E} = 4\pi\rho = \frac{c\,(\nabla\chi \operatorname{rot} \mathbf{B})}{8\pi\sigma} \tag{3.3}$$

Using the equation of continuity and Ohm's law we find that the time for the charge in (3.3) to build up is $(1+\chi)^{1/2}/4\pi\sigma$. Thus the equations of quasistationary electrodynamics (2.7) will be justified, i.e., the displacement current can be neglected, if

$$t_0 \gg \frac{(1+\chi)^{1/2}}{4\pi\sigma} \tag{3.4}$$

where t_0 is the characteristic time of the process.

We note that the analogy with diamagnetism is not complete. If the turbulent region is bounded by a vacuum, the following situation can arise: σ decreases to zero in the boundary layer, but χ is constant in this layer.

Then, instead of (2.9) and (3.2), we have

$$\operatorname{rot} \mathbf{B} = \frac{4\pi}{c} \frac{\sigma \mathbf{E}}{1+\chi}, \quad \mathbf{B}_{n_1} = \mathbf{B}_{n_2}, \quad \mathbf{B}_{t_1} = \mathbf{B}_{t_2}$$

when the microscopic magnetic permeability is unity.

4. The Two-Dimensional Case. It is of interest to consider the idealized two-dimensional case $v_z = 0$, $B_z = 0$, $\partial/\partial z = 0$, since it can be solved by an independent method [2]. Here the equation for the vector potential is completely analogous to the familiar heat equation for a fluid. This makes it possible to test the method described above.

Instead of (2.2) we obtain, in this case,

$$\mathbf{B}^{(2n)} = \frac{p}{2} \int_0^t \exp\,[\nu_m k^2 (t_1 - t)] \int d\mathbf{k}_1 \,[\mathbf{k}\,[(\mathbf{k}-\mathbf{k}_1)\, \varphi\,(\mathbf{k}-\mathbf{k}_1)\, \mathbf{B}^{(2n-2)}(\mathbf{k}_1, t_1)]]\, dt_1$$

$$+\frac{p}{2}\int_0^t \exp[\nu_m k^2 (t_1 - t)] \int d\mathbf{k}_1 [\mathbf{k}[(\mathbf{k}-\mathbf{k}_1)\varphi(\mathbf{k}_1)\mathbf{B}^{(2n-2)}(\mathbf{k}-\mathbf{k}_1, t_1)]]\, dt_1$$

$$p = \frac{m}{2}\int A(k)\, d\mathbf{k}$$

$$\mathbf{k} = \{k_x, k_y, 0\}, \quad \mathbf{k}_1 = \{k_{1x}, k_{1y}, 0\}, \quad \mathbf{B}^{(f)} = \{B_x^{(f)}, B_y^{(f)}, 0\}$$

B(k, t) satisfies the equation

$$\frac{\partial \mathbf{B}(\mathbf{k}, t)}{\partial t} + \nu_m k^2 \mathbf{B}(\mathbf{k}, t) = \frac{p}{2}\int [\mathbf{k}[\mathbf{k}\varphi(\mathbf{k}-\mathbf{k}_1)\mathbf{B}(\mathbf{k}_1, t)]]\, d\mathbf{k}_1$$

The equation

$$\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} = -\operatorname{rot} \nu_m \operatorname{rot}\left(1 + \frac{\chi}{2}\right)\mathbf{B} \tag{4.1}$$

describes the diffusion of the field when $\mu = (1 + \frac{1}{2}\chi)^{-1}$. We write the equations

$$\operatorname{rot}\mathbf{B} = \frac{4\pi}{c}\frac{\mathbf{E}\sigma}{1 + \frac{1}{2}\chi} - \frac{1}{2}\frac{[\nabla\chi\mathbf{B}]}{1 + \chi/2}, \quad \operatorname{rot}\mathbf{H} = \frac{4\pi}{c}\sigma\mathbf{E}$$

Although we are concerned here with a uniform electrical conductivity, the boundary value problem in the two-dimensional case is similar to that in the three-dimensional problem. $\mathbf{B}_t$ and $\operatorname{rot}_t \mathbf{B}$ are discontinuous at the boundary but the magnitude of the discontinuity is different. No space charge appears in the two-dimensional problem since the condition

$$\operatorname{div}\mathbf{j} = \frac{\partial}{\partial z} j_z = 0$$

is satisfied automatically.

5. Discussion of the Results. At time t = 0 let a weak magnetic field be impressed on a high-conductivity fluid. In addition suppose turbulence is induced in part of the fluid. Let $\mathbf{v} = 0$ and $\mathbf{B} = \mathbf{B}(\mathbf{r}, 0)$ at t = 0. **B** is continuous and $\operatorname{rot}_n \mathbf{B} = 0$ at the fluid-vacuum boundary. We do not interest ourselves in the question of how and for how long the turbulence is induced. We assume that at time t_1 a stationary state is reached, i.e., the statistical characteristics of the velocity become independent of time. At this instant diamagnetic properties appear and the field begins to be expelled from the turbulent region. The characteristic time of expulsion is

$$t_2 = \frac{L_1^2}{\nu_m(1+\chi)}$$

Here L_1 is a dimension of the turbulent region. The problem of the damping of the total field is now reduced to the problem of finding the eigenfunctions and eigenvalues of (2.4) by substituting $\mathbf{B} = \mathbf{A}(\mathbf{r}) \exp \times (-\gamma t)$ [5]. To find the approximate value of the minimum γ, and thereby the characteristic damping time of the field, it is sufficient to use for **E** the equation obtained from (2.7).

$$\frac{1}{\nu_m(1+\chi)^{1/2}}\frac{\partial \mathbf{E}}{\partial t} = -\operatorname{rot}(1+\chi)^{1/2}\operatorname{rot}\mathbf{E} \tag{5.1}$$

Equation (5.1) is true for all space if it is assumed that σ goes to zero continuously at the fluid-vacuum boundary. We multiply (5.1) scalarly by **E** and integrate over all space, taking account of the fact that $\mathbf{E}_t$ is continuous at the boundary of the turbulent region. The boundary conditions for $\operatorname{rot}_t \mathbf{E}$ are obtained from (3.2)

$$\frac{1}{2}\frac{d}{dt}\int \frac{E^2}{\nu_m(1+\chi)^{1/2}}\, d\mathbf{r} = -\int (1+\chi)^{1/2}(\operatorname{rot}\mathbf{E})^2\, d\mathbf{r} \tag{5.2}$$

Using (5.2) and the boundary conditions it is easy to estimate the damping time of the field

$$t_3 = L^2/\nu_m \tag{5.3}$$

We recall that (5.3) agrees with the damping time of the field in a solid conductor where L is a dimension of the whole fluid. It is obvious that (5.2) and (5.3) are valid if the dimensions of the nonturbulent part of the fluid are not too small in comparison with L_1, i.e., if $L_2^2 > L_1^2(1 + \chi)^{-1}$, where L_2 is the smallest diameter of the nonturbulent part of the fluid. Otherwise the magnetic field is not completely expelled and the damping of the total field occurs for a time t_2.

It is easy to see that $t_3 \gg t_2 \gg l/v$, where l and v are respectively a representative dimension and a characteristic velocity of the pulsations. Consequently neglecting the correlation time is justified.

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